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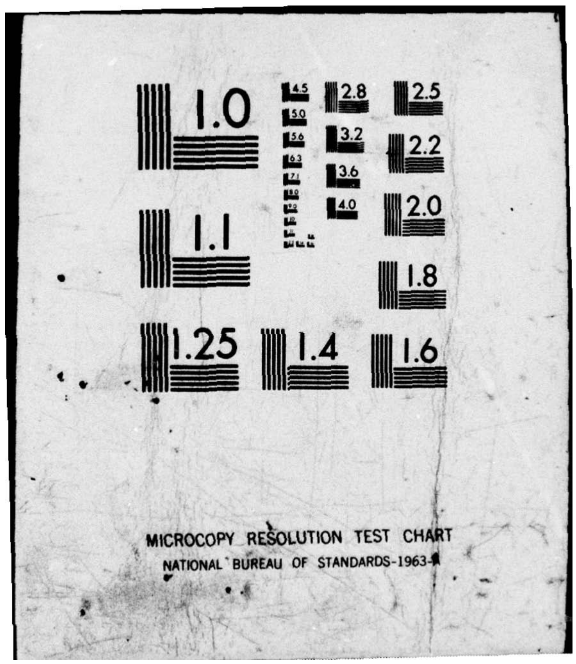
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A HETEROGENEOUS ARRIVAL AND SERVICE QUEUEING LOSS MODEL

by
SIMSON FOND
and
SHELDON M. ROSS

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ABSTRACT

Consider a single server exponential queueing loss system in which the arrival and service rates alternate between the pairs (λ_1,μ_1) and (λ_2,μ_2) , spending an exponential amount of time with rate $c\alpha_i$ in (λ_i,μ_i) , $i = 1,2$. It can be ~~is~~ shown that if all arrivals finding the server busy are lost then the percentage of arrivals lost is a decreasing function of c . This is in line with a general conjecture ~~of Ross [1]~~ to the effect that the "more nonstationary" a Poisson arrival process is then the greater the average customer delay (in infinite capacity models) or the greater the percentage of lost customers (in finite capacity models). We also study the limiting cases when c approaches 0 or infinity.

A HETEROGENEOUS ARRIVAL AND SERVICE QUEUEING LOSS MODEL

by

Simson Fond and Sheldon M. Ross

0. INTRODUCTION

This paper is a continuation of a study of queueing models with non-stationary Poisson arrivals begun in [1], where it was conjectured, and verified in a special case, that a queueing system with nonstationary Poisson arrivals will lead to larger average customer delays than would a similar model having stationary Poisson arrivals with the same average arrival rate. In order to further investigate this conjecture we consider a single server loss system that oscillates between two feasible levels denoted by 1 and 2. When the system is at level i $(i = 1,2)$ the arrival process is a Poisson process with rate λ_i and the service times are exponential random variables with rate μ_i. The time interval during which the system functions at level i is also an exponential random variable with rate $c\alpha_i$ where c is a constant, i.e. the persistence of the system at any level is governed by a random mechanism: if the system is functioning at level i it tends "to jump" to the alternative level with Poisson rate $c\alpha_i$.

We suppose that an arriving customer will only enter the system if the server is free when he arrives. Let $L(c)$ denote the proportion of customers that are lost to the system. In the following section we show that

$L(c)$ **is decreasing and convex in** c .

It should be noted that the (time) average arrival and service rates, call them $\bar{\lambda}$ and $\bar{\mu}$ are given by

$$\bar{\lambda} = \frac{\lambda_1\alpha_2 + \lambda_2\alpha_1}{\alpha_1 + \alpha_2} \; , \; \bar{\mu} = \frac{\mu_1\alpha_2 + \mu_2\alpha_1}{\alpha_1 + \alpha_2}$$

and are thus independent of c . The purpose of the constant c is that it regulates how fast the system changes levels; thus the larger c is then in some sense "the more stationary the process is." Indeed as c approaches infinity the system converges to a stationary one.

1. THE LOSS FUNCTION $L(c)$

The system can be analyzed as a continuous time Markov process with states $\{(m,i) \mid m = 0,1 \text{ and } i = 1,2\}$, where m denotes the number of customers in the system, and i denotes the level of the system. The transition probabilities are stationary and satisfy the forward Kolmogorov differential equations. Moreover, for all (m,i), the limiting probabilities, call them P_{mi}, exist and are independent of the initial state. The set $\{P_{mi}\}$ satisfies the following balance equations

$$(\lambda_1 + c\alpha_1)P_{01} = \mu_1 P_{11} + c\alpha_2 P_{02} \tag{1a}$$

$$(\mu_1 + c\alpha_1)P_{11} = \lambda_1 P_{01} + c\alpha_2 P_{12} \tag{1b}$$

$$(\lambda_2 + c\alpha_2)P_{02} = \mu_2 P_{12} + c\alpha_1 P_{01} \tag{2a}$$

$$(\mu_2 + c\alpha_2)P_{12} = \lambda_2 P_{02} + c\alpha_1 P_{11} \tag{2b}$$

with

$$P_{01} + P_{11} + P_{02} + P_{12} = 1 . \tag{3}$$

Let $L(c)$ denote the proportion of customers lost to the system. Since

$$\bar{\lambda} L(c) = \lambda_1 P_{11} + \lambda_2 P_{12}$$

we can calculate $L(c)$ by finding P_{11} and P_{12}. Before doing that let us note that the proportion of time the system is in level 1 is

$$P_{01} + P_{11} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \tag{4}$$

which can be obtained either by adding (1a) and (1b) together and substituting (3), or by considering the system as an alternating renewal process. Similarly,

$$P_{02} + P_{12} = \frac{\alpha_1}{\alpha_1 + \alpha_2} . \tag{5}$$

To solve for P_{11} and P_{12}, the easiest way is to put (1a), (1b) in a matrix form as follows

$$\begin{bmatrix} (\lambda_1 + c\alpha_1) & -\mu_1 \\ -\lambda_1 & (\mu_1 + c\alpha_1) \end{bmatrix} \begin{bmatrix} P_{01} \\ P_{11} \end{bmatrix} = \begin{bmatrix} c\alpha_2 P_{02} \\ c\alpha_2 P_{12} \end{bmatrix} . \tag{6}$$

Similarly, for (2a), (2b),

$$\begin{bmatrix} (\lambda_2 + c\alpha_2) & -\mu_2 \\ -\lambda_2 & (\mu_2 + c\alpha_2) \end{bmatrix} \begin{bmatrix} P_{02} \\ P_{12} \end{bmatrix} = \begin{bmatrix} c\alpha_1 P_{01} \\ c\alpha_1 P_{11} \end{bmatrix} . \tag{7}$$

Putting (6) and (7) together yields

$$\begin{bmatrix} (\lambda_1 + c\alpha_1) & -\mu_1 \\ -\lambda_1 & (\mu_1 + c\alpha_1) \end{bmatrix} \begin{bmatrix} (\lambda_2 + c\alpha_2) & -\mu_2 \\ -\lambda_2 & (\mu_2 + c\alpha_2) \end{bmatrix} \begin{bmatrix} P_{02} \\ P_{12} \end{bmatrix} = \begin{bmatrix} c^2\alpha_1\alpha_2 P_{02} \\ c^2\alpha_1\alpha_2 P_{12} \end{bmatrix} . \tag{8}$$

From the first row of (8) we obtain

$$[c(\alpha_1\lambda_2 + \alpha_2\lambda_1) + \lambda_2(\lambda_1 + \mu_1)]P_{02} = [c(\alpha_1\mu_2 + \alpha_2\mu_1) + \mu_2(\lambda_1 + \mu_1)]P_{12} .$$

Therefore,

$$\frac{P_{12}}{P_{02}} = \frac{c(\alpha_1\lambda_2 + \alpha_2\lambda_1) + \lambda_2(\lambda_1 + \mu_1)}{c(\alpha_1\mu_2 + \alpha_2\mu_1) + \mu_2(\lambda_1 + \mu_1)} \,. \tag{9}$$

Hence, by (5) and (9),

$$P_{12} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{c(\alpha_1\lambda_2 + \alpha_2\lambda_1) + \lambda_2(\lambda_1 + \mu_1)}{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

and

$$P_{02} = \frac{\alpha_1}{\alpha_1 + \alpha_2} \cdot \frac{c(\alpha_1\mu_2 + \alpha_2\mu_1) + \mu_2(\lambda_1 + \mu_1)}{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} \,.$$

Due to the symmetry of the equations (1a), (1b) and (2a), (2b), we see that

$$P_{11} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{c(\alpha_1\lambda_2 + \alpha_2\lambda_1) + \lambda_1(\lambda_2 + \mu_2)}{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

$$P_{01} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{c(\alpha_1\mu_2 + \alpha_2\mu_1) + \mu_1(\lambda_2 + \mu_2)}{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} \,.$$

Thus we have

$$\bar{\lambda}L(c) = \frac{c(\alpha_1\lambda_2 + \alpha_2\lambda_1)^2 + \lambda_1^2\alpha_2(\lambda_2 + \mu_2) + \lambda_2^2\alpha_1(\lambda_1 + \mu_1)}{(\alpha_1 + \alpha_2)\{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)\}} \,.$$

Differentiation yields that

$$\bar{\lambda}L'(c) = \frac{-\alpha_1\alpha_2(\lambda_1\mu_2 - \lambda_2\mu_1)^2}{(\alpha_1 + \alpha_2)\{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)\}^2}$$

and

$$\bar{\lambda}L''(c) = \frac{2\alpha_1\alpha_2(\lambda_1\mu_2 - \lambda_2\mu_1)^2[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)]}{(\alpha_1 + \alpha_2)\{c[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)] + (\lambda_1 + \mu_1)(\lambda_2 + \mu_2)\}^3} .$$

There are 2 cases to consider

Case 1:

$\lambda_1\mu_2 - \lambda_2\mu_1 = 0$, i.e., the traffic intensities λ_1/μ_1 and λ_2/μ_2 are equal, say to ρ .

In this case $L'(c) = 0$, and thus $L(c)$ is independent of the value c . Moreover we have simple solutions for the P_{mi}'s in this case, namely

$$P_{01} = \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{1}{1 + \rho}$$

$$P_{11} = \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{\rho}{1 + \rho}$$

$$P_{02} = \frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{1}{1 + \rho}$$

$$P_{12} = \frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{\rho}{1 + \rho} .$$

Hence, P_1 , the proportion of time the system is busy is

$$P_1 \equiv P_{11} + P_{12} = \frac{\rho}{1 + \rho} ,$$

and P_0 , the proportion of time the system is empty is

$$P_0 \equiv P_{01} + P_{02} = \frac{1}{1 + \rho} .$$

In terms of P_0 and P_1, the system functions as an ordinary M/M/1 loss system with traffic intensity ρ. The loss function is found to be

$$L(c) = \frac{\rho}{1+\rho} .$$

<u>Case 2</u>:

$$\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0 .$$

In this case $L'(c) < 0$ and $L''(c) > 0$. Hence $L(c)$ is a decreasing convex function of the value c.

Therefore, if the ratio of the time the system stays at each level is fixed, then the faster the system alternates between these two levels, the better the system is (in terms of the loss function).

2. EXTREME CASES

We have shown that $L(c)$ is a strictly decreasing function of c when the traffic intensities λ_1/μ_1 and λ_2/μ_2 are not equal. Now let us study the two extreme cases: (1) $c \to \infty$, i.e. the system alternates extremely fast between level 1 and level 2 or equivalently, the mean time the system stays at each level approaches 0; (2) $c \to 0$, i.e. the system alternates extremely slowly between level 1 and level 2 or, equivalently, the mean time the system stays at each level is becoming infinitely large.

Case 1: $c \to \infty$

$$\bar{\lambda} \lim_{c\to\infty} L(c) = \frac{(\alpha_1\lambda_2 + \alpha_2\lambda_1)^2}{(\alpha_1 + \alpha_2)[\alpha_1(\lambda_2 + \mu_2) + \alpha_2(\lambda_1 + \mu_1)]}$$

implying that

$$\lim_{c\to\infty} L(c) = \frac{\bar{\lambda}}{\bar{\mu} + \bar{\lambda}} .$$

Furthermore, the proposition of time the system is busy can be obtained by

$$P_1 = \lim_{c\to\infty} P_{11} + \lim_{c\to\infty} P_{12} = \frac{\bar{\lambda}}{\bar{\mu} + \bar{\lambda}}$$

and the proportion of time the system is idle is

$$P_0 = \lim P_{01} + \lim P_{02} = \frac{\bar{\mu}}{\bar{\mu} + \bar{\lambda}} .$$

Thus, the limiting system is equivalent to a no queue allowed M/M/1 system with constant arrival rate $\bar{\lambda}$ and service rate $\bar{\mu}$.

Since $L(c)$ is decreasing, the value $\frac{\bar{\lambda}}{\bar{\mu} + \bar{\lambda}}$ is the smallest value the system can achieve for the loss function.

<u>Case 2</u>: $c \to 0$

$$\bar{\lambda} \lim L(c) = \frac{\lambda_1^2\alpha_2(\lambda_2 + \mu_2) + \lambda_2^2\alpha_1(\lambda_1 + \mu_1)}{(\alpha_1 + \alpha_2)(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

$$= \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{\lambda_1^2}{\lambda_1 + \mu_1} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\lambda_2^2}{\lambda_2 + \mu_2}$$

and the proportion of time the system is busy is

$$P_1 = \lim_{c \to 0} P_{11} + \lim_{c \to 0} P_{12} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{\lambda_1}{\mu_1 + \lambda_1} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\lambda_2}{\mu_2 + \lambda_2}$$

the proportion of time the system is idle is

$$P_0 = \lim_{c \to 0} P_{01} + \lim_{c \to 0} P_{02} = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\mu_2}{\mu_2 + \lambda_2} .$$

Thus, the limiting system functions as the (time) average of two independent M/M/1 loss systems, one with arrival rate λ_1 and service rate μ_1 ; and the other with arrival rate λ_2 and service rate μ_2 .

3. RIGHT AND WRONG ARRANGEMENTS

Let us assume $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$, and compare the system R with levels (λ_1,μ_1) , (λ_2,μ_2) to the system W with levels (λ_1,μ_2) , (λ_2,μ_1) under the condition $\alpha_1 = \alpha_2$. In other words, the system R has the arrangement such that the server with slow service rate goes on the shift with the slow arrival rate and the person with the fast service rate goes on the shift with the fast arrival rate. The system W is arranged the other way around. If we denote the loss functions of the system R and W by L_R and L_W respectively, then a simple algebraic computation yields that $L_R(c) < L_W(c)$, and so the system R is better than the system W in the sense of loss function.

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